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DUALITY BETWEEN DETERMINISM AND STOCHASTICS AT SYSTEMS OF LINEAR EQUATIONS

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Summary

The present contribution derives and explains a general relation which furnishes a stochastical interpretation of a system of linear equations. The system of linear equations is interpreted as a transport of infinitely many chaotically fluctuating particles. These particles, which are called *stochants* in future, have the property to possess only a purely real or a purely imaginary part. They pass orientated graphes of a *stochant model* of the linear system with a probability which is proportional to the absolute value of the corresponding coefficient in the matrix of the system, and they change the phase by a multiple of $\pi/4$ according to the sign of the matrix element.

Since each noisy electrical network corresponds to a system of linear equations, it follows that the relations in the electrical network can also be described and interpreted by such chaotic motions of stochants. Thereby the sources of the electrical signal in the electrical network correspond to the sources of stochants in the presented stochant model, the transfer functions between sources and sinks of signals in the electrical network correspond to the probabilities in the stochant model with which a stochant of a certain source reaches a certain sink.

Finally the idea of the stochant model is explained with the aid of concrete examples.

1. Introduction

If a system of linear equations is given by

$$
\begin{array}{lllll}
a_{11}x_1 & + a_{12}x_2 & \cdots & + a_{1n}x_n & = y_1 \\
a_{21}x_1 & + a_{22}x_2 & \cdots & + a_{2n}x_n & = y_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n1}x_1 & + a_{n2}x_2 & \cdots & + a_{nn}x_n & = y_n
\end{array}
\qquad (1)
$$

or in matrix form

$$
(\mathbf{A}_{nn})(\mathbf{X}_n) = (\mathbf{Y}_n)\,, \qquad (2)
$$

then its solution by Cramer's rule

$$
x_i = y_1 \frac{|A_{1i}|}{|A|} + y_2 \frac{|A_{2i}|}{|A|} + \ldots + y_n \frac{|A_{ni}|}{|A|} = \sum_{j=1}^{n} y_j \frac{|A_{ji}|}{|A|}
$$

(3)

has a concrete physical meaning in the field of electrical engineering. Here, linear electrical networks are described by systems of linear equations, and a_{ij} in (1) are the impedances Z_{ij} resp. the admittances Y_{ij} of the network, x_i are the electrical signals resulting in the linear network (voltage drops U_i or branch currents I_i), and y_j are the electromotive forces (voltage sources e_m and/or current sources) placed in the network, refer to Fig. 1.

Therefore the notion signal transfer function T_{ij} for the quotient of x_i by y_j

$$T_{ij} = \frac{x_i}{y_j} = \frac{(-1)^{i+j} \det_{ji}\left\{ \mathbf{A}_{nn} \right\}}{\det\left\{ \mathbf{A}_{nn} \right\}} \tag{4}$$

Notice the order of indices at $\det_{ji}$ and at T_{ij} in (4) . The determinante in (4) $\det_{ji}$ {A} is the value of the determinante of the matrix ($\mathbf{A}_{nn}$), if its i-th column and j-th row are deleted, (hence $\det_{ji}$ {A} is of rank
n - 1). The signal transfer function T_{ij} connects the electromotive forces of the j-th row to the signal of the i-th column in (1).

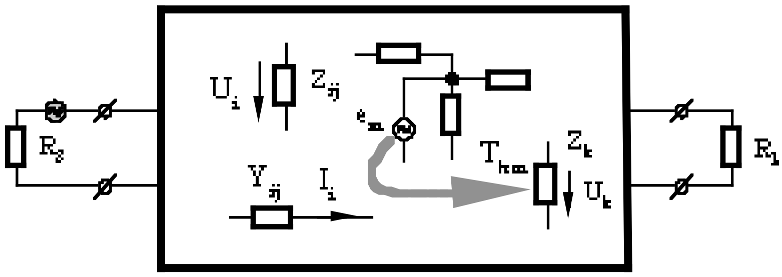

Fig.1 *Transport of electrical energy from the source e_m to the load Z_k*

The transfer function T_{ij} specifies which component of the signal x_i generates the electromotive force y_j , if all the other electromotive forces are zero. The signal x_i itself is by relation (3) the superposition of all components which generate the individual electromotive forces via their corresponding signal transfer function

$$x_i = T_{i1} y_1 + T_{i2} y_2 + \dots + T_{in} y_n = \sum_{j=1}^{n} T_{ij} y_j \tag{5}$$

The transfer functions of the electrical network have a picturesque meaning in the Mason-Model [1]. If (E_{nn}) is the identity matrix and if a matrix (B_{nn}) is constructed by the following principle

$$(\mathbf{B}_{nn}) = (\mathbf{E}_{nn}) - (\mathbf{A}_{nn}) \tag{6}$$

then (A_{nn}) satisfies

$$(\mathbf{A}_{nn}) = (\mathbf{E}_{nn}) - (\mathbf{B}_{nn}) . \tag{7}$$

Then we also get

$$(\mathbf{A}_{nn})(\mathbf{X}_n) = (\mathbf{X}_n) - (\mathbf{B}_{nn})(\mathbf{X}_n). \qquad (8)$$

This leads to the analytic reason of the Mason-Model in [1], because of (2) and (8) it yields

$$(\mathbf{X}_n) = (\mathbf{B}_{nn})(\mathbf{X}_n) + (\mathbf{Y}_n) \ . \qquad (9)$$

If one expands the equation (9) for one element of the vector (X_n) one gets the expression

$$x_k = b_{k1}x_1 + \quad \dots \quad + b_{kj}x_j \dots + b_{kn}x_n + y_k \qquad (10)$$

2. The stochant model

Let first the deduction of the stochant model be given by a linear system of degree 2

$$\left| \begin{array}{l} a_{11}\,x_1 + a_{12}\,x_2 = y_1 \\ a_{21}\,x_1 + a_{22}\,x_2 = y_2 \end{array} \right. \qquad . \qquad (11)$$

The two transfer functions for x_1 with (3) and (5) are for n = 2

$$x_1 = \frac{a_{22}}{a_{11}a_{22} - a_{12}a_{21}}\,y_1 + \frac{(-a_{12})}{a_{11}a_{22} - a_{12}a_{21}}\,y_2 \quad . \qquad (12)$$

One can change the coefficients a_{kj} in (12) by the coefficients b_{kj} of the matrix **($\mathbf{B}_{nn}$)** . If one notices (7) , this transforms the expression (12) into

$$x_1 = \frac{1}{(1 - b_{11})\left[1 - \frac{b_{12}b_{21}}{(1 - b_{11})(1 - b_{22})}\right]}\,y_1 + \frac{b_{12}}{(1 - b_{11})(1 - b_{22})\left[1 - \frac{b_{12}b_{21}}{(1 - b_{11})(1 - b_{22})}\right]}\,y_2 \ .$$

(13)

For $b_{k\,k} \neq 1$ the relation

$$\frac{1}{1-b} = \sum_{n=0} b^n \quad \text{if} \quad |b| < 1 \quad \text{or} \quad \frac{1}{-b\left(1-\frac{1}{b}\right)} = \frac{1}{-b} \sum_{n=0} \frac{1}{b^n} \quad \text{if} \quad |b| > 1 \qquad (14)$$

can be applied to (13) . A multiple application of (14) in (13) leads to a new expression for x_1

$$x_1 = \left(\sum_{n=0} b_{1\,1}^n \right) \left\{ \sum_{m=0} \left[b_{1\,2}\, b_{2\,1} \left(\sum_{s=0} b_{1\,1}^s \right) \left(\sum_{t=0} b_{2\,2}^t \right) \right]^m \right\} y_1$$

$$+\, b_{1\,2} \left(\sum_{p=0} b_{1\,1}^p \right) \left(\sum_{q=0} b_{2\,2}^q \right) \left\{ \sum_{r=0} \left[b_{1\,2}\, b_{2\,1} \left(\sum_{u=0} b_{1\,1}^u \right) \left(\sum_{v=0} b_{2\,2}^v \right) \right]^r \right\} y_2 \qquad (15).$$

The case $b_{k\,k} = 1$ does not play any role in our situation, because in this case it is not necessary in (13) to factor out ($1 - b_{k\,k}$) and develop it into a geometric series in (14).

Then one can repeat the transformation from (12) to (15) , which was performed for x_1 , also for x_2 . For x_2 one gets the following expression

$$x_2 = b_{2\,1} \left(\sum_{m=0} b_{1\,1}^m \right) \left(\sum_{n=0} b_{2\,2}^n \right) \left\{ \sum_{p=0} \left[b_{1\,2}\, b_{2\,1} \left(\sum_{q=0} b_{1\,1}^q \right) \left(\sum_{r=0} b_{2\,2}^r \right) \right]^p \right\} y_1$$

$$+\left(\sum_{s=0} b_{22}^{s}\right)\left\{\sum_{t=0}\left[b_{12}\, b_{21}\left(\sum_{u=0} b_{11}^{u}\right)\left(\sum_{v=0} b_{22}^{v}\right)\right]^{t}\right\} y_2 \qquad (16)$$

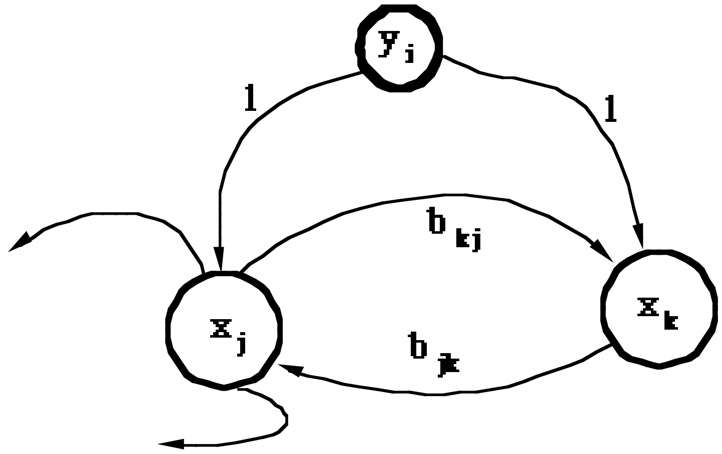

Fig.2 *The transfer of signals in the sense of the stochant model*

The expressions (15) and (16) establish a relation of a new quality which represents the functions x_1 and x_2 as the result of infinitely many chaotic fluctuations. Thereby the elements b_{kj} and b_{jk} of the matrix **(B)** are parameters in the deterministic point of view. At the same time the b_{kj} and b_{jk} are probabilities of the chaotic transfer of the system from the status k to the status j and vice versa. The proof of the stochant model for $n > 2$ is given by induction starting from $n = 2$.

This allows to solve each system of linear equations successively with decreasing inaccuracy by a Monte Carlo method. Furthermore it is remarkable that one uses only the two basic operations addition and subtraction to invert the matrix (A) into (A)$^{-1}$.

Examples for these solutions of linear systems are given

3. The Electrical Network with Noise

The presented duality can be perfectly applied to the noisy electrical network. The network is described by a system of linear equations. Therefore it can be either a chaotically fluctuating system or a system with deterministic relations between the individual electrical terms. Hence the transfer of electrical energy and the corresponding transfer function can not only be seen as an algebraic solution of the corresponding linear system but just as well as a stochastic transfer of infinitely many and infinitely weak elements of energy, ref. to Fig. 3.

Therefore there exists a duality between determinism and stochastics at systems of linear equations.

[1] S. J. Mason, 1953, "Feedback Theory - Some Properties of Signal Flow Graphs", Proceedings of the IRE, Vol.41 No.9 pp.1144-1156 Sept. 1953

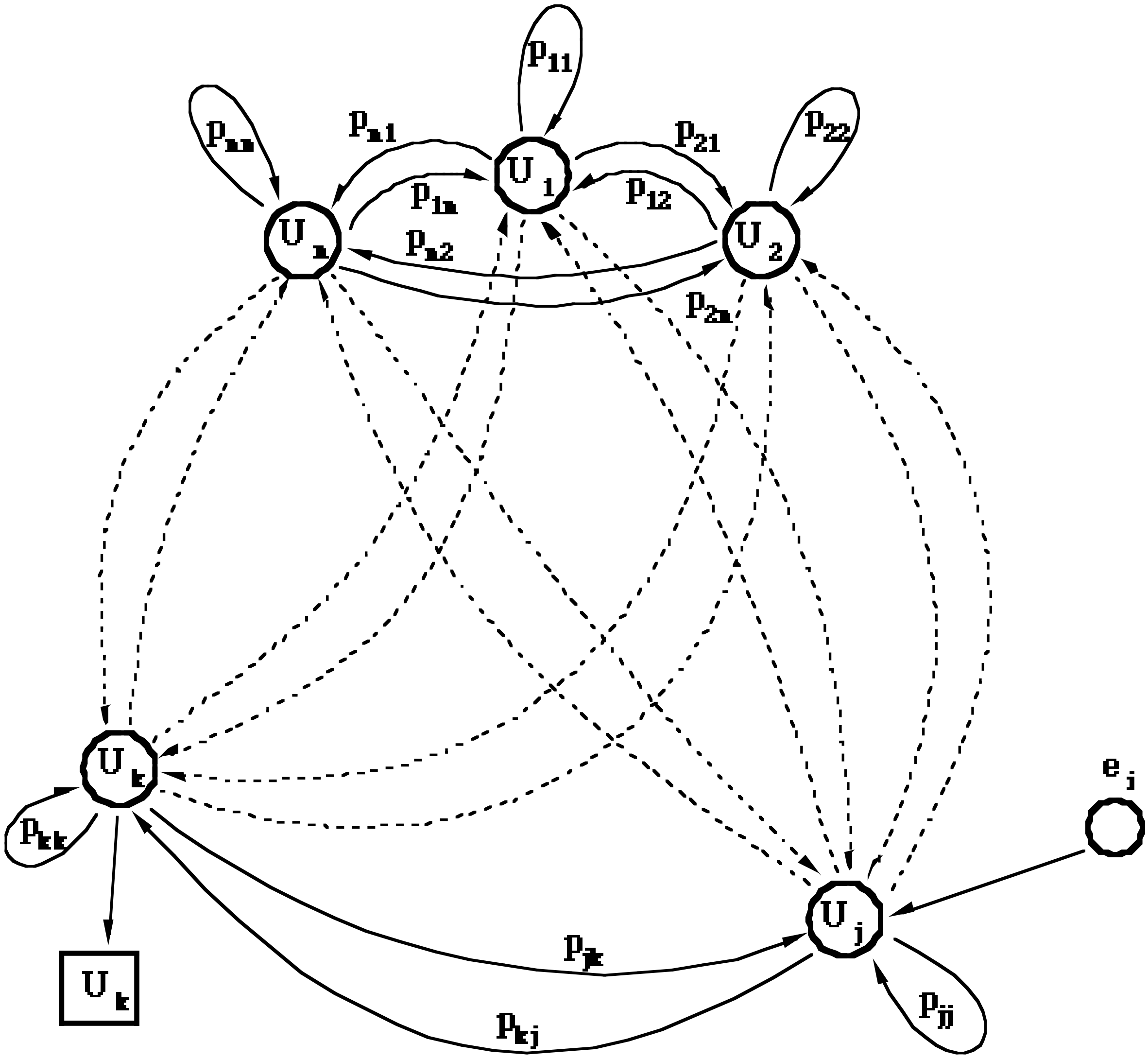

Fig.3 *Stochastical presentation of the linear system of equations for a electrical network* $(A)_{nn}(U)_n = (e)_n$